Performance Prediction

Chapter 3

1.0 Governing Simulation Equations –

When determining the time history of velocity and altitude, we first derive an equation for acceleration. For pure vertical linear motion along the rocket's axis, the acceleration is derived from the sum of the forces acting parallel to this axis and through the rocket's center of gravity. These forces are thrust 'T', weight 'W', and drag 'D'. Thrust is assumed to act in the direction of flight. Weight and drag oppose the direction of flight. A simple equation for acceleration 'a' is accomplished by referring to the diagram below, summing the forces, and solving for acceleration.

$$\sum Forces = Ma = T - D - W$$

Where,

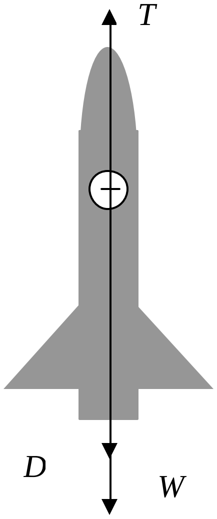
$M = Mass$

$a = Acceleration$

$T = Thrust$

$D = Drag$

$W = Weight$



$$\longrightarrow \quad a = \frac{1}{M}\left(T - D - W\right)$$

The weight of an object is a force. This force is the product of the acceleration of gravity, denoted by 'g', and the object's mass.

$$W = Mg$$

$$\longrightarrow \quad M = \frac{W}{g}$$

Another relation for acceleration is obtained by replacing M with W/g, and rearranging the terms.

$$a = \frac{g}{W}\left(T - D - W\right) \quad \longrightarrow \quad a = g\left(\frac{T}{W} - 1\right) - \frac{D}{W}$$

Recall that the rocket's drag can be expressed as,

$$D = \frac{1}{2} r V^2 C_D p R^2 \,,$$

Where,

r = *Density of Air, lbs-s^2 /ft^4*
V = *Velocity, ft/s*
C_D = *Rocket's drag coefficient*
p = *Constant PI ~ 3.14159*
R = *Rocket's maximum body radius, ft*

The drag equation is combined with the acceleration equation to give a relation for acceleration as a function of velocity.

$$a = g\left(\frac{T}{W} - 1\right) - \frac{rC_D pR^2 g}{2W}V^2$$

A useful form of the acceleration equation is obtained by expressing it in terms of two constants and velocity.

$$\longrightarrow a = k_1 + k_2V^2, \qquad \textbf{\textit{Equation 3.1}}$$

Where,.

$$k_1 = g\left(\frac{T}{W} - 1\right)$$
$$k_2 = -\frac{rC_D pR^2 g}{2W}$$

As noted above, acceleration is a function of thrust, weight, drag-coefficient, acceleration of gravity and air density. Thrust and weight are functions of time as noted in the previous chapters. Drag coefficient varies with speed, or more correctly, dynamic pressure 'q'. The acceleration of gravity and air density change with altitude above sea level. For the purposes of calculating velocity and altitude with time, the analysis is broken into small increments of time. The drag-coefficient, weight, acceleration of gravity, thrust, and air density are assumed constant over each increment of time. Time is the independent variable, and because thrust and weight are known functions of time, their average values over each increment of time are known. To minimize error, the average values of thrust and drag coefficient are used in each incremental calculation. It is assumed that the remaining variables are nearly constant over each increment and equal to their initial values corresponding to the time at the beginning of the increment. A small change in velocity 'dV' is related to an increment in time 'dt' as follows:

$$dV = adt = \left(k_1 + k_2V^2\right)dt$$

$$\longrightarrow dt = \frac{dV}{a} = \frac{dV}{\left(k_1 + k_2V^2\right)}$$

Let subscript '1' relate to values at the beginning of a given time increment, and subscript '2' relate to values at the end of that time increment. Then the time increment 'dt' can be integrated as follows:

$$\int_{t_1}^{t_2} dt = \int_{V_1}^{V_2} \frac{dV}{\left(k_1 + k_2V^2\right)}$$

$$\longrightarrow (t_2 - t_1) = \begin{cases} \dfrac{1}{\sqrt{k_1 k_2}}\left\{\tan^{-1}\left(\dfrac{V_2\sqrt{k_1 k_2}}{k_1}\right) - \tan^{-1}\left(\dfrac{V_1\sqrt{k_1 k_2}}{k_1}\right)\right\}, \mathit{For}\ k_1 k_2 \geq 0 \\ \dfrac{1}{\sqrt{-k_1 k_2}}\left\{\tanh^{-1}\left[\dfrac{V_2\sqrt{-k_1 k_2}}{k_1}\right] - \tanh^{-1}\left[\dfrac{V_1\sqrt{-k_1 k_2}}{k_1}\right]\right\}, \mathit{For}\ k_1 k_2 \leq 0 \end{cases}$$

Whenever thrust is greater than weight, k_1 is greater than zero. Whenever k_1 is greater than zero, the quantity k_1k_2 is less than zero. In this situation, the second of the above two equations for time increment $(t_2\text{-}t_1)$ is used in deriving the relation for velocity at the end of that time increment. After performing some algebraic manipulation to the second of the above two equations, the following expression for the final velocity of the time increment is derived:

Whenever $T > W$,

$$\longrightarrow V_2 = \frac{k_1}{\sqrt{-k_1 k_2}}\tanh\left\{\tanh^{-1}\left[\frac{V_1\sqrt{-k_1 k_2}}{k_1}\right] + (t_2 - t_1)\sqrt{-k_1 k_2}\right\}, \quad \textbf{\textit{Equation 3.2}}$$

Where,

$$k_1 = g\left(\frac{T}{W} - 1\right)$$

$$k_2 = -\frac{rC_D pR^2 g}{2W}$$

$V_1 = \mathit{Velocity\ at\ start\ of\ time\ increment}$

$t_1 = \mathit{Time\ at\ start\ of\ time\ increment}$

$t_2 = \mathit{Time\ at\ end\ of\ time\ increment}$

$\tanh = \textbf{\textit{Hyperbolic}}\ \mathit{tangent}$

$\tanh^{-1} = \textbf{\textit{Inverse hyperbolic}}\ \mathit{tangent}$

Whenever thrust is less than weight, k_1 is less than zero. Whenever k_1 is less than zero, the quantity k_1k_2 is greater than zero. In this situation, the first equation for time increment $(t_2\text{-}t_1)$ is used in deriving a relation for the velocity at the end of that time increment. After performing some algebraic manipulation to the first equation, the following expression for the final velocity of the time increment is derived:

Whenever $T < W$,

$$\longrightarrow V_2 = \frac{k_1}{\sqrt{k_1 k_2}}\tan\left\{\tan^{-1}\left[\frac{V_1\sqrt{k_1 k_2}}{k_1}\right] + (t_2 - t_1)\sqrt{k_1 k_2}\right\}, \qquad \textbf{\textit{Equation 3.3}}$$

Where,

$\tan = \mathit{Tangent}$

$\tan^{-1} = \mathit{Inverse\ Tangent}$

$$k_1 = g\left(\frac{T}{W} - 1\right)$$
$$k_2 = -\frac{rC_D pR^2 g}{2W}$$
$$V_1 = Velocity\ at\ start\ of\ time\ increment$$
$$t_1 = Time\ at\ start\ of\ time\ increment$$
$$t_2 = Time\ at\ end\ of\ time\ increment$$

The rocket's gain in altitude 'dh' over time increment 'dt' is equal to Vdt. Likewise, as previously noted the change in velocity 'dV' over time increment 'dt' is equal to adt. By algebraic manipulation of these to relations and the substitution of Equation 3.1 for acceleration, a differential form of gain in altitude, as a function of velocity, can be derived as follows:

$$Given:\ \frac{dh}{dt} = V \longrightarrow dh = Vdt, \quad and \quad \frac{dV}{dt} = a \longrightarrow dt = \frac{dV}{a}.$$

$$Therefore,\ dh = \frac{V}{a}dV = \frac{V}{\left(k_1 + k_2V^2\right)}dV$$

Now, the differential form of gain in altitude can be integrated with respect to velocity to give an equation for altitude change as a function of velocities 'V_1' and 'V_2'.

$$\int_{h_1}^{h_2} dh = \int_{V_1}^{V_2} \frac{V}{\left(k_1 + k_2V^2\right)}dV$$

$$\left(h_2 - h_1\right) = \frac{1}{2k_2}\left[\ln\left(k_1 + k_2V_2^2\right) - \ln\left(k_1 + k_2V_1^2\right)\right] = \frac{1}{2k_2}\left[\frac{\ln\left(k_1 + k_2V_2^2\right)}{\ln\left(k_1 + k_2V_1^2\right)}\right]$$

$$\longrightarrow h_2 = \frac{1}{2k_2}\left[\ln\left(\frac{k_1 + k_2V_2^2}{k_1 + k_2V_1^2}\right)\right] + h_1, \qquad \textbf{Equation 3.4}$$

Where,

$$h_1 = Altitude\ at\ start\ of\ time\ increment$$
$$k_1 = g\left(\frac{T}{W} - 1\right)$$
$$k_2 = -\frac{rC_D pR^2 g}{2W}$$
$$V_1 = Velocity\ at\ start\ of\ time\ increment$$
$$V_2 = Velocity\ at\ end\ of\ time\ increment$$

Equations 3.1 through 3.4 along with knowledge of the rocket's drag, thrust and weight can be used to calculate the rocket's time history of acceleration, velocity and altitude. The following two examples describe different approaches to simulating the rocket's performance.

2.0 Example 1 –

The above equations are extremely powerful when used for purposes of preliminary design and parametric ("what if") analyses. These equations can be used in hand calculations or simple spreadsheets to better understand flight physics or narrow in on a design space when laying out a new rocket design.

The following example is an attempt to understand flight physics before setting up a matrix of parameters to define a design space (i.e.; sensitivity analysis). Let us consider the design of a two-stage rocket with a total impulse within the limits of the Tripoli Level I classification. The objective is to maximize altitude within the thrust limits of two existing commercial engines, the Aerotech I161W for the first stage boost and the Aerotech H123W for the second stage boost. These engines were chosen because of their combination of medium thrust level at long duration. For a given impulse level, high thrust levels will generate large accelerations and velocities, and in turn large drag. Recall that drag is proportional to the square of the speed. Drag is driven by the rocket's geometry, thrust (recall base drag) and speed. On the other hand, speed contributes to the rocket's inertia. The greater the inertia the greater the potential for higher altitudes. Typically, we think of the rocket's mass, geometry, thrust and stability when designing for maximum altitude. However, when considering the opposing phenomena of drag and inertia, both being functions of velocity, it would be reasonable to expect that there exists an optimum coast duration between first and second stage burns. Therefore, an additional parameter to consider for designing for maximum altitude of two-stage rockets is this coast duration between burns. The purpose of the following example is to determine if it is reasonable to consider the sensitivity of maximum altitude to coast duration between burns.

This example will be divided into four phases, initial burn, coast, separation and second stage burn, and second stage coast. We will refer to our design as Donner Und Blitz.

2.1– Initial Burn Phase

The liftoff weight of Donner Und Blitz is 3.539 lbs. The weight of fuel burned is 0.413 lbs. Therefore, the average rocket weight during this phase is 3.3325 lbs.

$$\longrightarrow W = 3.539 - \frac{0.413}{2} = 3.3325 \ lbs.$$

The I161W produces a total impulse of 78.684 lbs-secs. The duration of burn is 2.25 seconds. The average thrust during this phase is equal to the total impulse divided by the duration of burn.

$$\longrightarrow T = \frac{I}{t} = \frac{78.684}{2.25} = 34.97 \ lbs.$$

Without having knowledge of the detailed geometry, we will assume the average drag coefficient over the burn phase to be 0.4. We will also assume standard sea level conditions. Therefore the air density is assumed to have a value of 0.002378 lbs-sec^2/ft^4 and the gravity constant a value of 32.174 ft/sec^2. The rocket's body radius is 1.1085 inches or 0.092375 feet. The initial values of time, velocity, and altitude above ground are zero. We now have enough information to determine the rocket's performance during the initial burn phase.

The acceleration constants are calculated as follows:

$$k_1 = g\left(\frac{T}{W} - 1\right) = 32.174\left(\frac{34.97}{3.3325} - 1\right) = 305.4478$$

$$k_2 = -\frac{rC_D pR^2 g}{2W} = -\frac{(.002378)(0.4)p\,(.092375)^2(32.174)}{2(3.33245)} = -0.000123$$

The velocity at the end of the burn phase is calculated using Equation 3.2. Substituting in the appropriate values gives:

$$V_2 = \frac{(305.4)}{\sqrt{-(305.4)(-.000123)}} \tanh\left\{\tanh^{-1}(0) + (2.25 - 0)\sqrt{-(305.4)(-.000123)}\right\} = 646.76\, ft/s$$

The altitude at the end of this phase is calculated using Equation 3.4. Substituting in the appropriate values gives:

$$h_2 = \frac{1}{2(-.000123)} \ln\left\{\frac{305.4 + (-.000123)(646.76)^2}{305.4 + (-.000123)(0)^2}\right\} + 0 = 749.8\, ft$$

2.2– First Coast Phase

The weight for this phase is a constant because no fuel is being burned and the lower and upper stages have not separated. Therefore the weight is the liftoff weight minus the total fuel weight burned during the first burn phase.

$$\longrightarrow W = 3.539 - 0.413 = 3.126\, lbs.$$

The thrust is zero during this phase, and consequently the average rocket drag coefficient increases due to base drag effects. We assume the drag coefficient increases to 0.55.

The initial time (t_1), velocity (V_1), and altitude (h_1) of this phase are equal to the final values $(t_2$, V_2, and $h_2)$ of the previous phase.

The acceleration constants are calculated as follows:

$$k_1 = g\left(\frac{T}{W} - 1\right) = 32.174\left(\frac{0}{3.126} - 1\right) = -32.174 \approx -32$$

$$k_2 = -\frac{rC_D pR^2 g}{2W} = -\frac{(.002378)(0.55)p\,(.092375)^2(32.174)}{2(3.126)} = -0.000180$$

The velocity at the end of the first coast phase is calculated using Equation 3.3. The time at the end of this phase is a variable we must provide. Recall that the purpose of this example is to determine the sensitivity of final altitude to the coast duration between burn phases. For the calculation given below we will set t_2 equal to 10.25 seconds, giving a coast duration of 8 seconds.

$$V_2 = \frac{(-32)}{\sqrt{(-32)(-.00018)}} \tan\left\{ \tan^{-1}\left[\frac{(646)\sqrt{(-32)(-.00018)}}{-32} \right] + (10.25 - 2.25)\sqrt{(-32)(-.00018)} \right\} = 170.35\, ft/s$$

The altitude at the end of this phase is calculated using Equation 3.4.

$$h_2 = \frac{1}{2(-.00018)} \ln\left\{ \frac{-32 + (-.00018)(170.35)^2}{-32 + (-.00018)(646.76)^2} \right\} + 749.8 = 3682\, ft$$

3.3 - Second Burn Phase

Separation of the upper stage from the lower stage is assumed instantaneous. The initial weight of for this phase is 2.126 lbs. The weight of fuel burned is 0.276 lbs. Therefore, the average rocket weight during this phase is 1.988 lbs.

$$\longrightarrow W = 2.126 - \frac{0.276}{2} = 1.988\ lbs.$$

The H123W produces a total impulse of 51.706 lbs-secs. The duration of burn is 2.55 seconds. The average thrust during this phase is equal to the total impulse divided by the duration of burn.

$$\longrightarrow T = \frac{I}{t} = \frac{51.706}{2.55} = 20.28\ lbs.$$

We will assume the average drag coefficient over the second burn phase to be 0.35.

The initial time (t_1), velocity (V_1), and altitude (h_1) of this phase are equal to the final values $(t_2$, V_2, and $h_2)$ of the previous coast phase.

The acceleration constants are calculated as follows:

$$k_1 = g\left(\frac{T}{W} - 1\right) = 32.174\left(\frac{20.28}{1.988} - 1\right) = 296.04$$

$$k_2 = -\frac{rC_D pR^2 g}{2W} = -\frac{(.002378)(0.35)p\,(.092375)^2(32.174)}{2(1.988)} = -0.000181$$

The velocity at the end of the burn phase is calculated using Equation 3.2. The time t_2 used in this equation is the time t_1 plus the burn time.

$$V_2 = \frac{296}{\sqrt{-(296)(-.000181)}} \tanh\left\{ \tanh^{-1}\left[\frac{(170)\sqrt{-(296)(-.000181)}}{296} \right] + (12.8 - 10.25)\sqrt{-(296)(-.000181)} \right\} = 792.34\, ft/s$$

The altitude at the end of this phase is calculated using Equation 3.4.

$$h_2 = \frac{1}{2(-.000181)} \ln\left\{\frac{296.04 + (-.000181)(792.34)^2}{296.04 + (-.000181)(170.35)^2}\right\} + 3682 = 4970.3\,ft$$

3.4 – Final Coast Phase

The weight for this phase is a constant because all the fuel has been expended. The weight is simply the zero fuel weight of the upper stage.

$$\longrightarrow W = 1.85\ lbs.$$

The thrust is zero during this phase, and consequently the average rocket drag coefficient increases due to base drag effects. We assume the drag coefficient increases from the 0.35 during the second burn phase to 0.5 for the final coast phase.

The initial time (t_1), velocity (V_1), and altitude (h_1) of this phase are equal to the final values $(t_2$, V_2, and $h_2)$ of the second burn phase.

The acceleration constants are calculated as follows:

$$k_1 = g\left(\frac{T}{W} - 1\right) = 32.174\left(\frac{0}{1.85} - 1\right) = -32.174 \approx -32$$

$$k_2 = -\frac{rC_D pR^2 g}{2W} = -\frac{(.002378)(0.5)p\,(.092375)^2(32.174)}{2(1.85)} = -0.000277$$

The final coast phase ends with the rocket decelerating to zero velocity $(V_2 = 0)$ at flight apex. The initial velocity (V_1) is equal to the final velocity (V_2) of the second burn phase. Therefore, the initial and final velocities are known for the final coast phase. The unknowns are the duration of coast $(t_2 - t_1)$ and final altitude (H_2). Using the following equation previously derived for $(t_2 - t_1)$, and the appropriate variable values, the duration of coast and total flight time can be calculated.

$$(t_2 - t_1) = \frac{1}{\sqrt{k_1 k_2}}\left\{\tan^{-1}\left(\frac{V_2\sqrt{k_1 k_2}}{k_1}\right) - \tan^{-1}\left(\frac{V_1\sqrt{k_1 k_2}}{k_1}\right)\right\},\ For\ k_1k_2\ ³\ 0$$

$$(t_2 - t_1) = \frac{1}{\sqrt{(-32)(-.000277)}}\left\{\tan^{-1}\left(\frac{(0)\sqrt{(-32)(-.000277)}}{-32}\right) - \tan^{-1}\left(\frac{(792)\sqrt{(-32)(-.000277)}}{-32}\right)\right\} = 12.38\ secs$$

$$\longrightarrow t_2 = t_1 + 12.38 = 12.8 + 12.38 = 25.18\ secs$$

From the above calculations, the coast duration to zero velocity is 12.38 seconds. This means that either a delay charge duration or an electronic timer should be set for a delay of about 12 seconds after the second stage burn has completed. The total flight time to apex is just over 25 seconds. The maximum flight altitude is calculated using equation 3.4. Substituting in the appropriate values gives:

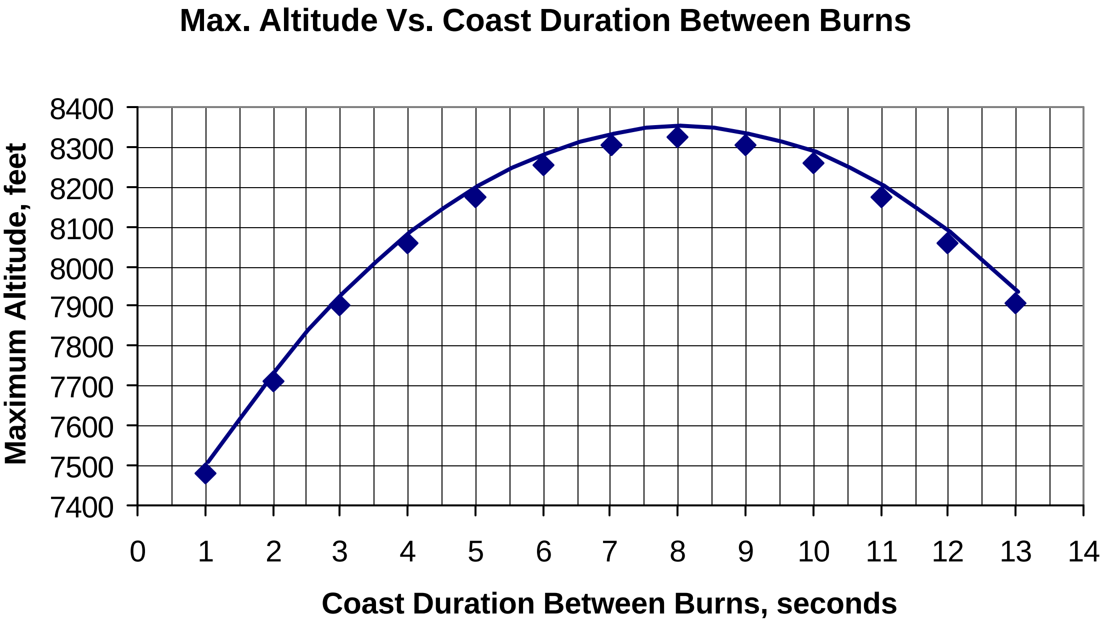
$$h_2 = \frac{1}{2(-.000277)} \ln \left\{ \frac{-32 + (-.000277)(0)^2}{-32 + (-.000277)(792)^2} \right\} + 4970 = 8322\ ft$$

A summary of the entire flight profile from liftoff to apex is given in the following table.

Performance Calculations for Donner Und Blitz

Time (seconds)	Velocity (ft/secs)	Altitude (ft)
0	0.00	0
2.25	646.76	749.8
10.25	170.35	3682.0
12.80	792.34	4970.3
25.18	0.00	8322.4

Recall that the purpose of this example is to determine if it is reasonable to consider the sensitivity of maximum altitude to coast duration between burns. The above calculations were repeated several times for different coast duration values between burns. The graph below illustrates the results.



The results, as illustrated in the graph, suggest that altitude attainable is very sensitive to coast duration between burns. The fact that there is an optimum value that produces a maximum, suggests that the opposing phenomena of drag and inertia are very important considerations in rocket performance. That is, an optimum combination of shape and mass exists that maximizes altitude for given total impulse. Higher thrust and lighter weight is not always better.

3.0 Example 2 –

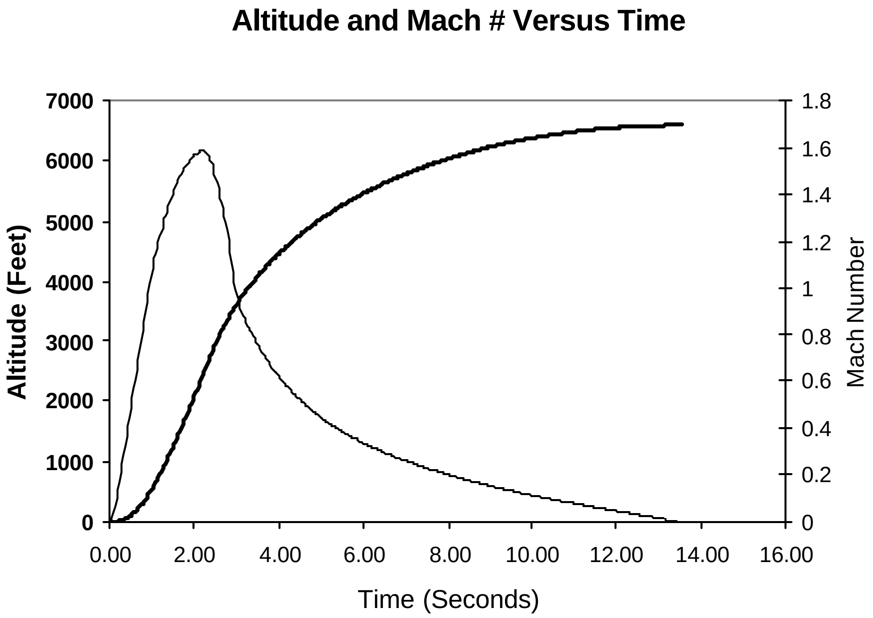
During a rocket's flight the drag, weight, and thrust change continuously with time and altitude. The airflow characteristics such as density and viscosity change with altitude; and because altitude changes with time then it is reasonable to expect that the airflow characteristics change with time. Likewise the gravity constant changes with altitude; and hence, with time.

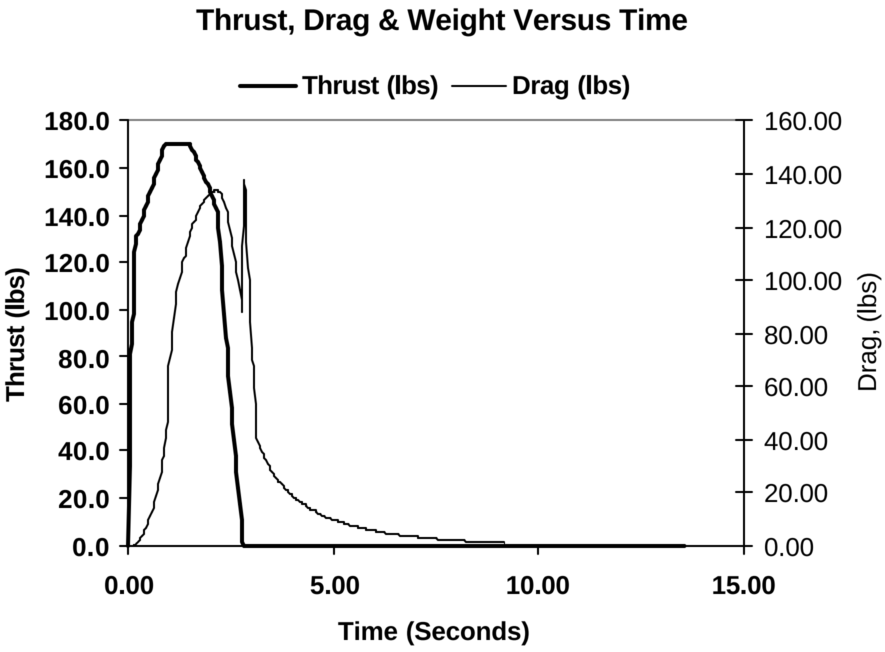
We can approximate the continuous time history of a rocket's flight by selecting sufficiently small changes in time $(Dt = t_2 - t_1)$ such that the characteristics of the fastest changing flight-independent variable is accurately accounted for. Usually the fastest changing flight-independent variable is thrust for a non air-breathing rocket. Dt may not be a constant, but may vary throughout a simulation. A value of Dt on the order of 0.01-seconds or less is reasonable. It is necessary for the first Dt $(Dt = t_2 - t_1,$ *where* $t_1 = 0.0$ *and* $t_2 = Dt)$ of the simulation to be sufficiently large such that the average thrust-to-weight ratio is greater than 1.0, and the resulting value of acceleration 'a' (Equation 1) is greater than zero.

It is up to the author of the simulation to define how he or she would characterize the drag coefficient, the gravity constant, air density and viscosity over a given Dt. Thrust and weight are known values of time; hence, their average values over Dt are known and should be used in the calculations. The initial values of drag coefficient, gravity constant, air density and viscosity are known at the start of time increment Dt, but their average values over Dt, are not known. It is often reasonable to assume that the initial values of drag coefficient, gravity constant, air density and viscosity are constant over the time increment Dt, without introducing much error into the simulation. At the end of the time increment, these variables are then recalculated and used in the next set of time increment calculations. If the analyst chooses to program the simulation in a language that supports looping, he or she may design the simulation to iterate on a given time increment. This allows for convergence on average values of drag coefficient, gravity constant, air density and viscosity, velocity and altitude over time increment Dt.

Utilizing the equations presented in this chapter as well as the previous chapters, a simulation program was developed to estimate the continuous time history of single stage rockets. The subject rocket for this example is called Terraphobic. A summary table of rocket characteristics and performance is given below, along with various time history graphs.

Rocket Name:	**Terraphobic**	
Length:	55.06	inches
Diameter:	5	inches
Impulse:	369.850	lbs-secs
Thrust Duration:	2.800	seconds
Launch Weight:	3.362	lbs
Burnout Weight:	2.752	lbs
Stability Margin @ Launch:	1.25	
Stability Margin @ Burnout:	1.70	
Number of Fins:	4	
Nose Design:	ogive	
Maximum Altitude:	6573	feet
Maximum Mach No.:	1.59	
Maximum Velocity:	1755	feet/sec
Maximum G-Loading:	42.90	g's
Total Flight Time To Apex:	13.50	seconds


Altitude and Mach # Versus Time
Altitude (Feet)
0
1000
2000
3000
4000
5000
6000
7000
Mach Number
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Time (Seconds)
0.00
2.00
4.00
6.00
8.00
10.00
12.00
14.00
16.00


Thrust, Drag & Weight Versus Time
Thrust (lbs)
Drag (lbs)
Thrust (lbs)
0.0
20.0
40.0
60.0
80.0
100.0
120.0
140.0
160.0
180.0
Drag, (lbs)
0.00
20.00
40.00
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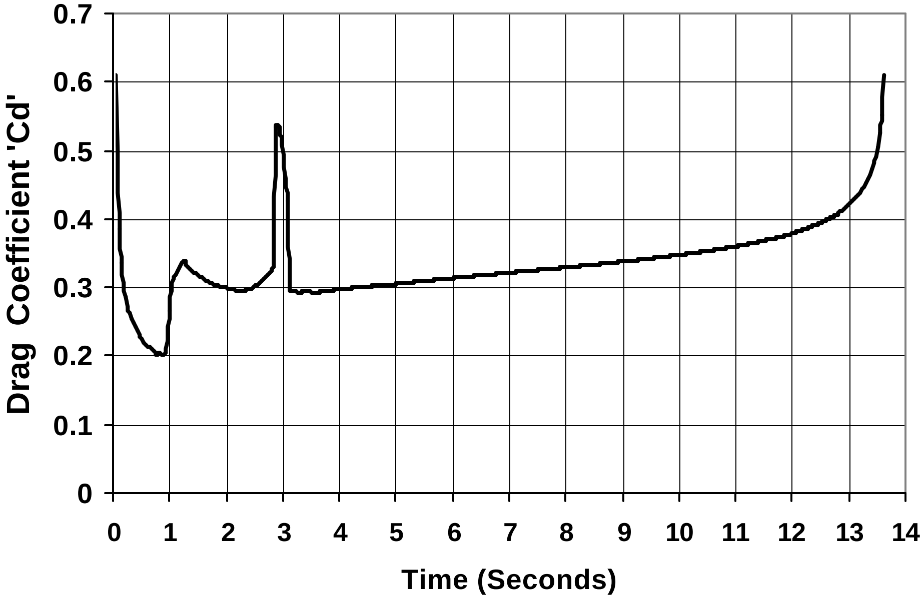
80.00
100.00
120.00
140.00
160.00
Time (Seconds)
0.00
5.00
10.00
15.00


Drag Coefficient 'Cd'
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
Time (Seconds)