Estimate of Weight Variation with Burn Time

Chapter 2

We will formulate an estimate of rocket weight variation with burn time. Recall that as the engine burns fuel, the rocket's weight decreases. As the weight decreases, the Center-of-Gravity (C.G.) shifts forward, improving its static and dynamic stability characteristics. It is often of interest to know the rocket's weight as a function of time. We can use this information for detailed performance and stability analyses.

The derivation to follow is not entirely accurate and may have a few critical technical flaws. Feedback is welcome in the form of constructive criticism to either improve the method or replace it all together.

Rocket engine propulsion consists of a mass momentum contribution as well as differential pressure. As amateur rocket scientists, we typically have knowledge of thrust versus time characteristics of our engines as well as initial engine weight and weight at burnout. The change in rocket weight is related to the mass flow of the exhaust gas. This accounts for the mass momentum contribution to thrust, but not differential pressure effects. It is assumed that the time history of the mass flow rate is identical in form to the thrust time history, assuming all thrust contributors are identical in shape as well. We can now proceed with the derivation.

"For every action there exists an opposite and equal reaction". This is the fundamental law of nature that governs rocket propulsion, known as Newton's Third Law of Motion. Sir Isaac Newton, not to be confused with the more familiar Fig, is credited with this discovery. However, there has been debate that Mr. Newton may have stolen his discovery from Chicken Little, who exclaimed, "The sky is falling, the sky is falling!" Mathematically, thrust can be expressed by two terms, the first describing the mass momentum contribution, and the second describing the pressure differential at the nozzle exit. In equation form, we say:

$$T = \frac{\dot{W}}{g} V_e + (P_e - P_{am}) A_e = \text{Thrust in lbs.}$$

Where,

$\dot{W} =$ Mass flow rate, lb./s
$g =$ Acceleration due to gravity = 32.174 ft/s^2
$V_e =$ Exhaust velocity at nozzle exit, ft/s
$P_e =$ Pressure at nozzle exit, lb/in^2
$P_{am} =$ Ambient pressure of free stream, lb/in^2
$A_e =$ Cross sectional area at nozzle exit, in^2

Now, to apply the previously discussed assumption we assume that P_e is equal to P_{am}. This is the same as saying that the pressure at the nozzle exit is the same as the surrounding ambient pressure. If the fixed nozzle area is properly designed for a constant chamber pressure, this condition may be obtainable and the flow is considered fully expanded. However, the chamber pressure will probably vary somewhat, and P_e will not equal P_{am}, resulting in a negative or positive contribution to thrust. Nevertheless, we will put our trust in the engine designers, and assume that P_e is nearly equal to P_{am}, or that their difference times the nozzle exit area is negligible relative to the mass momentum. The thrust equation reduces to,

$$T \approx \frac{\dot{W}}{g} V_e \text{, = Thrust in lbs.}$$

Where,

$\dot{W} =$ Mass flow rate, lb./s
$g =$ Acceleration due to gravity = 32.174 ft/s^2
$V_e =$ Exhaust velocity at nozzle exit, ft/s

By rearranging the above equation, we can express the mass flow rate as a function of thrust and its dependency on time '*t*', as:

$$\dot{W}(t) \approx \frac{g}{V_e} T(t) \text{ = Mass flow rate, lb./s}$$

If we know the fuel mass flow rate as a function of time, then we can calculate the rocket's weight as a function of time. The rocket's weight at time '*t*' is simply the rocket's initial weight minus the total weight of the fuel burned at time '*t*'. To calculate the burned fuel at time '*t*', we simply integrate (add up) the contribution of fuel burned prior to time '*t*'. For a small change in time '*dt*', we have a small amount of fuel burned dW_f. In equation form we say,

$$dW_f = \dot{W}(t)dt \text{ = Small value of fuel burned, in lbs.}$$

The total fuel weight burned at time '*t*' equals the sum of the dWs over the duration of time '*t*'. This operation is called Integration. Mathematically we can describe the total fuel weight burned at time '*t*' as:

$$W_f(t) = \int_0^t \dot{W}(t)dt = \frac{g}{V_e}\int_0^t T(t)dt$$

The fancy stuff after that weird looking 'S' symbol (integration symbol) means summing up the thrust contributions over time. Graphically, this is the area under the thrust versus time curve up to the time of interest '*t*'. Most computer programs approximate this operation by using simple discrete calculations; two of the most common are the "Trapezoidal Rule" and "Simpson's Rule". The "Trapezoidal Rule" is the least accurate but easiest to implement. For our purposes, the "Trapezoidal Rule" is adequate.

We are interested in knowing the rocket's weight as a function of time. As mentioned earlier, in order to calculate the rocket's weight at time '*t*', we must subtract the burned fuel weight at time '*t*'. Thus,

$$W(t) = W_o - W_f(t) = W_o - \frac{g}{V_e}\int_0^t T(t)dt$$

Where,

$W_o =$ Initial rocket weight with full fuel, lbs.

The velocity at the nozzle exit is assumed to be independent of time! This was done for the ease of equation derivation. The laws of Physics have been violated and we may have completely invalidated the analysis. It is left up to the more enlightened to debunk this approach. Nevertheless, we shall proceed and eliminate the gravity constant '*g*' and exit velocity '*Ve*' using a wave of the hand, some hocus pocus and a bit of logic. We know the weight of the rocket at burnout, or zero fuel weight. Let us designate it as

‘W_{bo}’. The engine manufacturers typically provide the total impulse of the engine ‘I’. Recall that the total impulse is defined as the average engine thrust times the total burn time. This is the same as the total area under the thrust versus time curve, which is the same as that strange looking equation evaluated at the burnout time. That is,

$$I = (T_{av})(t_{bo}) = \int_0^t T(t)dt = \text{Total engine impulse, lb.-s}$$

Where,

T_{av} = Average engine thrust, lbs.

t_{bo} = Time at burnout, s

For engine burnout, we can plug into the weight equation the rocket’s zero fuel weight and replace the funky looking integral operation with the engine’s total impulse. The results is,

$$W_{bo} = W_o - I\frac{g}{V_e} = \text{Burnout weight in lbs.}$$

We can rearrange the above equation to give,

$$\frac{g}{V_e} = \frac{(W_o - W_{bo})}{I}.$$

We substitute the above equation into the time dependent equation for rocket weight to give the final form,

$$W(t) = W_o - \frac{(W_o - W_{bo})}{I}\int_0^t T(t)dt$$

We will now work an example problem. The following data is for a scratch built rocket called Icarus.

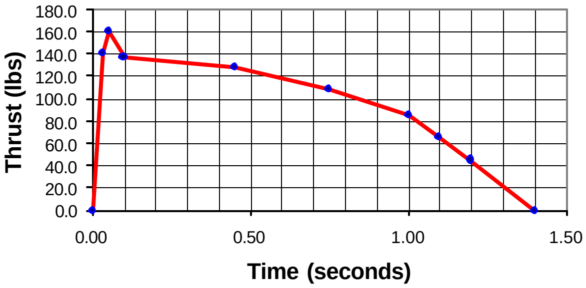
$W_o = 3.142$ lbs., Rocket weight at launch.

$W_{bo} = 2.532$ lbs., Rocket weight at burnout.

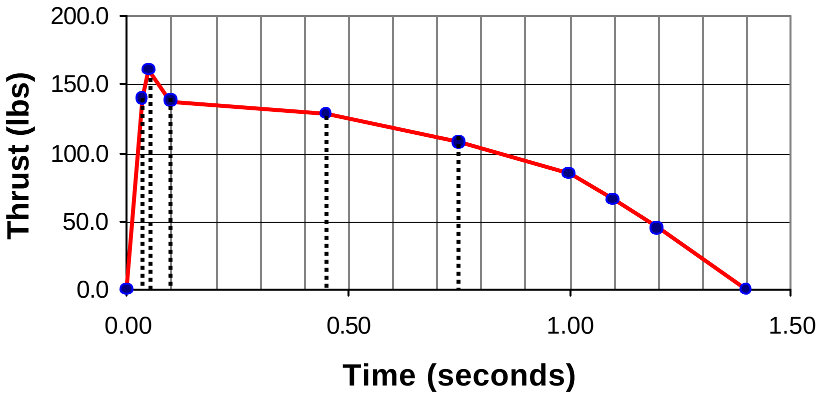
$I = 135.725$ lbs.-s, Total impulse of engine

$t_{bo} = 1.4$ s, time at engine burnout

Time (s)	Thrust (lbs.)
0.000	0
0.035	140
0.050	160
0.100	138
0.450	128
0.750	108
1.000	85
1.100	65
1.200	45
1.400	0



We will determine the rocket’s weight at time t = 0.75 seconds. Recall that the Integration function is simply the area under the thrust versus time curve, up to the time of interest. We can solve for this graphically using the “Trapezoidal Rule”. Graphically, this is the summation of the trapezoidal areas given by each time increment.



The area of each trapezoid is equal to its time increment multiplied by its average thrust. In mathematical terms, the integral is approximated as,

$$\int_0^t T(t)dt \approx \sum_i (t_{i+1} - t_i)\frac{(T_{i+1} + T_i)}{2}$$

For our problem, the calculation is:

$$\int_0^t T(t)dt \approx (0.035 - 0.0)\frac{(140+0)}{2} + (0.05 - 0.035)\frac{(160+140)}{2}$$
$$+ (0.1 - .05)\frac{(138+160)}{2} + (0.45 - 0.10)\frac{(128+138)}{2} + (0.75 - 0.45)\frac{(108+128)}{2}$$
$$\approx 94.1$$

Finally, the rocket weight at t = 0.75 seconds is given by:

$$\longrightarrow W(t) = W_o - \frac{(W_o - W_{bo})}{I}\int_0^t T(t)dt = 3.142 - \frac{(3.142 - 2.532)}{135.725}(94.1) = 2.719 \text{ lbs.}$$

The following graph illustrates the complete time history of the rocket's weight during engine burn.